THIN DOMAINS WITH EXTREMELY HIGH OSCILLATORY BOUNDARIES

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Abstract. In this paper we analyze the behavior of solutions of the Neumann problem posed in a thin domain of the type $R^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1), -\epsilon\, b(x_1) < x_2 < \epsilon\, G(x_1, x_1/\epsilon^\alpha)\}$ with $\alpha > 1$ and $\epsilon > 0$, defined by smooth functions $b(x)$ and $G(x,y)$, where the function G is supposed to be $l(x)$-periodic in the second variable y. The condition $\alpha > 1$ implies that the upper boundary of this thin domain presents a very high oscillatory behavior. Indeed, we have that the order of its oscillations is larger than the order of the amplitude and height of R^ϵ given by the small parameter ϵ. We also consider more general and complicated geometries for thin domains which are not given as the graph of certain smooth functions, but rather more comb-like domains.

1. Introduction

In this paper, we analyze the behavior of the solutions of the Laplace equation with homogeneous Neumann boundary conditions

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = h^\epsilon & \text{in } R^\epsilon \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 & \text{on } \partial R^\epsilon \end{cases} \tag{1.1}$$

where N^ϵ is the unit outward normal to ∂R^ϵ and $h^\epsilon \in L^2(R^\epsilon)$. The domain R^ϵ is a two dimensional thin domain which presents a highly oscillatory behavior at the boundary. We will be able to consider two different types of thin domains, which will be clearly defined in Section 2. To make the ideas clear we will refer in this introduction to the first type: assume R^ϵ is given as the region between two functions, that is,

$$R^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -\epsilon\, b(x_1) < x_2 < \epsilon\, G_\epsilon(x_1)\} \tag{1.2}$$

where $b(\cdot)$ and $G_\epsilon(\cdot)$ are functions satisfying $0 < b_0 < b(\cdot) < b_1$, $0 \le G_\epsilon(\cdot) \le G_1$ for some fixed positive constants b_0, b_1 and G_1, independent of $\epsilon > 0$. Here, the function b, independent of ϵ, defines the lower boundary of the thin domain, and the function G_ϵ, dependent of ϵ, the upper boundary of R^ϵ. We will allow G_ϵ to present oscillations whose amplitude is larger than the order of compression of the thin domain. This is expressed by assuming that

$$G_\epsilon(x) = G(x, x/\epsilon^\alpha), \tag{1.3}$$

for some positive constant $\alpha > 1$. The function $G : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a positive smooth function, with $y \to G(x,y)$ periodic in y for fixed x with period $l(x)$.

Let us observe that our assumptions includes the case where the function G_ϵ presents a purely periodic behavior, for instance, $G_\epsilon(x) = 2 + \sin(x/\epsilon^\alpha)$. But it also considers the case where the function G_ϵ defines a thin domain where the oscillations period, the amplitude and the profile vary with respect to $x \in (0,1)$. The Figure 1 and 2 below illustrate kinds of thin domains that we are considering here.

Since the domain R^ϵ is thin, approaching the interval $(0,1)$, it is reasonable to expect that the family of solutions will converge to a function of just one variable and that this function will satisfy certain elliptic equation in one dimension with some boundary conditions.

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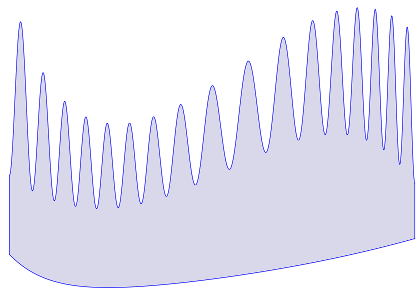

FIGURE 1. A thin domain with variable period, amplitude and profile.

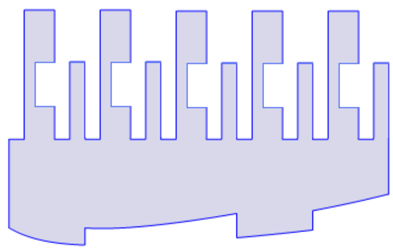

FIGURE 2. A comb-like thin domain.

It is known that if the domain does not present oscillations, that is $G_\epsilon(x) = G(x)$, with $0 < G_0 \leq G(\cdot) \leq G_1$ and $b(\cdot) \equiv 0$ the 1-dimensional limiting problem is given by

$$\begin{cases} -\dfrac{1}{G(x)}\Big(G(x)\, w_x\Big)_x + w = h, \text{ in } (0,1), \\ w_x(0) = w_x(1) = 0 \end{cases} \tag{1.4}$$

see for instance [10, 12]. Also, if we consider $b \equiv 0$, $G_\epsilon(x) = G(x, x/\epsilon^\alpha)$ for some $0 \leq \alpha < 1$, and if we assume that $G_\epsilon(\cdot) \to m(\cdot) \quad w - L^2(0,1)$ and $\frac{1}{G_\epsilon(\cdot)} \to k(\cdot) \quad w - L^2(0,1)$ (observe that $m(x)k(x) \geq 1$ a.e. and in general it is not true that $m(x)k(x) \equiv 1$), then the limit problem is

$$\begin{cases} -\dfrac{1}{m(x)}\left(\dfrac{1}{k(x)} w_x\right)_x + w = h, \text{ in } (0,1) \\ w_x(0) = w_x(1) = 0 \end{cases}$$

see [2] for details. Note that this case contains the previous one since we can recover the problem (1.4) assuming $\alpha = 0$.

Recently, we consider in [4, 5] a class of oscillating thin domain that cover the case $\alpha = 1$ with constant period l. Observe that this situation is very resonant since the height of the domain, the amplitude of the oscillations at the boundary and the period of the oscillations are of the same order ϵ. The limit problem for this case is

$$\begin{cases} -\dfrac{1}{s(x)}(r(x) w_x)_x + w = h(x), \quad x \in (0,1) \\ w'(0) = w'(1) = 0 \end{cases} \tag{1.5}$$

where

$$r(x) = \int_{Y^*(x)} \Big\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\Big\} dy_1 dy_2,$$

$$s(x) = |Y^*(x)|$$

and $X(x)$ is a convenient auxiliary harmonic function defined in the representative basic cell $Y^*(x)$, which depends on $G(x,\cdot)$, $x \in (0,1)$, and it is given by

$$Y^*(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < l, \quad 0 < y_2 < G(x, y_1)\}.$$

The restricted case where the function $G_\epsilon(x) = G(x/\epsilon)$ for some l-periodic smooth function G can be addressed by somehow standard techniques in homogenization theory, as developed in [6, 8, 13]. We refer to [3] for a complete analysis of this case for a semilinear parabolic problem.

In this work, we are interested in addressing the case $\alpha > 1$ in (1.3), where none of the techniques used to solve the previous ones apply. In particular, we do not use any extension operator for the convergence proof. Indeed, we will be able to show how the geometry of the boundary oscillations affect the limiting equation, see Theorem 2.1, Theorem 2.5. See also Corollary 2.3 for a very interesting interpretation of the limiting equation and to see how the geometry of the unit cell affect the limit equation in the case of periodic oscillations.

In Section 2 we give precise definitions of the two types of thin domains we are considering. One of them is the one described in this introduction. The other type is a "comb-like" thin domain, which can be visualize in Figure 2. We also state clearly the two main results we prove, Theorem 2.1 and Theorem 2.5.

The short Section 3 states a technical result which will be used later in the proof.

In Section 4 we analyze the type of thin domains which are given as a region between two graphs as in (1.2).

In Section 5 we analyze the other type of thin domains, that we have denoted as a "comb-like" thin domain.

We also would like to observe that although we will treat the Neumann boundary condition problem, we may also impose different conditions in the lateral boundaries of the thin domain R^ϵ, while preserving the Neumann type boundary condition in the upper and lower boundary. Indeed, we may consider conditions of the Dirichlet type, $w^\epsilon = 0$, or even Robin, $\frac{\partial w^\epsilon}{\partial N} + \beta w^\epsilon = 0$ in the lateral boundaries of the problem (1.1). The limit problem will preserve this boundary condition.

2. Basic facts, notation and main results

We will consider two different types of thin domains. One of them will be given as the region between the graphs of two functions and the other will consists of an autoreplicating structure with appropriate scaling rates which resembles a comb structure. We present now the main definitions, basic facts and results on both cases.

Type I. Thin domain as the region between two graphs. Let us consider a one parameter family of functions $G_\epsilon : (0,1) \to [0,\infty)$, $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$, and a function $b : (0,1) \mapsto (0,\infty)$. We will assume the following hypotheses on functions b and G_ϵ:

(H1) There exist two positive constants b_0, b_1 such that $0 < b_0 \le b(x) \le b_1$ for all $x \in (0,1)$ and the function b is piecewise C^1. .

(H2) The functions $G_\epsilon(\cdot)$ are of the type $G_\epsilon(x) = G(x, x/\epsilon^\alpha)$, with $\alpha > 1$, where the function

$$\begin{aligned} G : [0,1] \times \mathbb{R} &\longrightarrow [0, +\infty) \\ (x,y) &\longrightarrow G(x,y) \end{aligned} \tag{2.1}$$

is continuous in x, uniform in the second variable y, (that is, for each $\eta > 0$, there exists $\delta > 0$ such that $|G(x,y) - G(x',y)| \le \eta$ for all x, $x' \in [0,1]$, $|x - x'| < \delta$, and $y \in \mathbb{R}$). Moreover, we suppose $G(x,y) \ge 0$ is periodic in y, with a period $l(x)$ that may depend on the first variable, that is, $G(x, y + l(x)) = G(x,y)$. We also assume that $l(\cdot)$ is a continuous function with $0 < L' \le l(x) \le L$ for all $x \in [0,1]$.

We consider the highly oscillating thin domain R^ϵ, which is given as the region between the graphs of the two functions $\epsilon b(\cdot)$ and $\epsilon G_\epsilon(\cdot)$, that is

$$R^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -\epsilon\, b(x_1) < x_2 < \epsilon\, G_\epsilon(x_1)\}$$

and we investigate the behavior of the solutions of (1.1) as $\epsilon \to 0$.

The Figure 3 below, gives us an exemple of function G in a bounded open set.

Since $\alpha > 1$, we have that the upper boundary of this thin domain presents a extremely high oscillatory behavior. More precisely, the order of the oscillations is large than the order of the amplitude and height of the thin domain R^ϵ with respect to the small parameter ϵ. Also, we get more general perturbations assuming that the period l depends on variable $x \in (0,1)$.

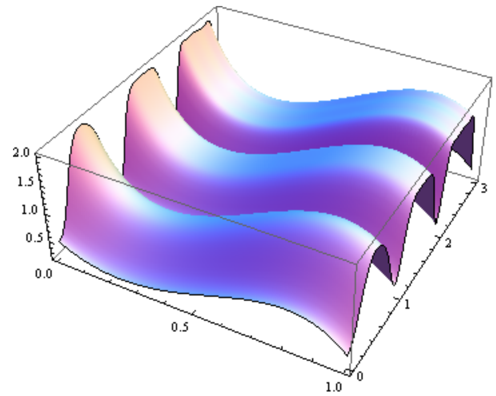

FIGURE 3. Graph of function G in $(0,1) \times (0,3L)$.

To study the convergence of the solutions of (1.1), we consider the equivalent linear elliptic problem

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\epsilon & \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1}\nu_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2}\nu_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases} \tag{2.2}$$

where $f^\epsilon \in L^2(\Omega^\epsilon)$ satisfies

$$\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \le C \tag{2.3}$$

for some $C > 0$ independent of ϵ, and now, $\nu^\epsilon = (\nu_1^\epsilon, \nu_2^\epsilon)$ is the outward unit normal to $\partial\Omega^\epsilon$, and $\Omega^\epsilon \subset \mathbb{R}^2$ is a highly oscillating domain given by

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -b(x_1) < x_2 < G_\epsilon(x_1)\}. \tag{2.4}$$

Note that the equivalence between (1.1) and (2.2) is easily obtained by changing the scale of the thin domain R^ϵ in the y-direction through the simple transformation $(x,y) \to (x, \epsilon y)$, (see [2, 10] for more details). Thus, we have a domain which is not thin anymore but presents very wild oscillatory behavior at the top boundary, although the presence of a high diffusion coefficient in front of the derivative with respect the second variable decreases the effect of the high oscillations.

We also mention the works [1, 7, 9] that analyse elliptic problems in domains related to Ω^ϵ but the fact that in our case we allow very high diffusion in the y-direction with distinct oscillations makes our analysis and results different from these other papers.

Now we are in contidion to state our main result whose proof will be presented in section 4.2.

Theorem 2.1. *Assume that $f^\epsilon \in L^2(\Omega^\epsilon)$ satisfies $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \le C$ and the function $\hat{f}^\epsilon(x) = \int_{-b(x)}^{G_\epsilon(x)} f(x,y)\,dy$ satisfies that $\hat{f}^\epsilon \rightharpoonup \hat{f}$, w-$L^2(0,1)$. Let u^ϵ be the unique solution of (2.2) and G_0 be the function given by*

$$G_0(x) = \min_{y \in \mathbb{R}} G(x,y) \ge 0. \tag{2.5}$$

Then, if $u_0(x_1)$ is the unique weak solution of the Neumann problem

$$\int_0^1 \Big\{ \Big(b(x) + G_0(x)\Big)\, u_x(x)\, \varphi_x(x) + p(x)\, u(x)\, \varphi(x) \Big\} dx = \int_0^1 \hat{f}(x)\, \varphi\, dx, \quad \forall \varphi \in H^1(0,1) \tag{2.6}$$

where $p(x)$ is the function defined as follows:

$$p(x) = \frac{1}{l(x)} \int_0^{l(x)} \Big(b(x) + G(x,y) \Big)\, dy = b(x) + \frac{1}{l(x)} \int_0^{l(x)} G(x,y)\, dy, \text{ for all } x \in (0,1),$$

we have

$$\|u^\epsilon - u_0\|_{L^2(\Omega^\epsilon)} \xrightarrow{\epsilon \to 0} 0.$$

Moreover, if we denote by $\Omega_0 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ -b(x_1) < x_2 < G_0(x_1)\} \subset \Omega_\epsilon$ then,

$$u_\epsilon \rightharpoonup u_0 \quad \omega - H^1(\Omega_0)$$

Remark 2.2. *The functions b and G_0 defined in (H1) and (2.5) respectively, are associated to the part of the domain Ω^ϵ that does not oscillate as the parameter ϵ goes to zero, and have an important role in the limit problem (2.6). Indeed, if we assume that the period, the amplitude and the profile of the domain are constant with respect to $x \in (0,1)$, we get the nice result announced below.*

Corollary 2.3. *If we have $G(x,y) = G(y)$ an L-periodic function with $\min_{y \in \mathbb{R}} G(y) = 0$ and $b(x) = b$ a constant function, then, the homogenized limit is given by the equation with constant coefficients:*

$$\begin{cases} -du'' + u = f, & (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{2.7}$$

where the diffusion coefficient is given by

$$d = \frac{b}{b + \frac{1}{L}\int_0^L G(y)\, dy} = \frac{Lb}{Lb + \int_0^L G(y)\, dy} = \frac{\textit{Area of the non oscillating part of the unit cell}}{\textit{Total area of the unit cell}}.$$

Remark 2.4. *In Corollary 2.3 the constant Lb represents the area of the unit cell which correspond to the part which is non oscillating and $Lb + \int_0^L G(y)\, dy$ represents the total area of the unit cell.*

Type II. Comb-like thin domain. We consider now another interesting type of thin domain. Consider

$$R^\epsilon = \mathrm{Int}\left(\overline{\mathrm{R}^\epsilon_- \cup \mathrm{R}^\epsilon_+}\right)$$

where

$$R^\epsilon_- = \{(x_1, x_2) \mid 0 < x_1 < 1, -\epsilon b(x_1) < x_2 < 0\},$$

with b given as in the previous case (see hypothesis (H1)), and

$$R^\epsilon_+ = \cup_{n=1}^{N_\epsilon} R^\epsilon_{n,+},$$

where

$$R^\epsilon_{n,+} = \{(nL\epsilon^\alpha + \epsilon^\alpha x_1, \epsilon x_2) \mid (x_1, x_2) \in Q_0\}$$

where $Q_0 \subset (0, L) \times (0, G)$ is a fixed Lipschitz domain satisfying the following:

(HQ) if $\Gamma_0 = \partial Q_0 \cap \{x_2 = 0\}$ and if we consider $e_1(Q_0)$ the first eigenvalue of the operator $-\Delta$ in Q_0 with homogeneous Dirichlet boundary condition in Γ_0 and homogeneous Neumann boundary condition in $\partial Q_0 \setminus \Gamma_0$, then $e_1(Q_0) > 0$.

Observe that if Q_0 is connected and $\Gamma_0 \neq \emptyset$ then (HQ) is satisfied. But there are cases where Q_0 is disconnected ant still (HQ) holds.

As we have done in the previous case, let us define $\Omega^\epsilon = \{(x_1, x_2) \mid (x_1, \epsilon x_2) \in R^\epsilon\}$ so that,

$$\Omega^\epsilon = \mathrm{Int}\left(\overline{\Omega_- \cup \Omega^\epsilon_+}\right),$$
$$\Omega_- = \{(x_1, x_2) \mid 0 < x_1 < 1, -b(x_1) < x_2 < 0\},$$
$$\Omega^\epsilon_+ = \cup_{n=1}^{N_\epsilon} \Omega^\epsilon_{n,+},$$

where

$$\Omega^\epsilon_{n,+} = \{(nL\epsilon^\alpha + \epsilon^\alpha x_1, x_2) \mid (x_1, x_2) \in Q_0\}.$$

We also consider the equivalent linear elliptic problem

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\epsilon \quad \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1}\nu_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2}\nu_2^\epsilon = 0 \quad \text{on } \partial\Omega^\epsilon \end{cases} \tag{2.8}$$

Under this conditions, we may get the following result.

Theorem 2.5. *Let u^ϵ be the unique solution of (2.2). Assume that $f^\epsilon \in L^2(\Omega^\epsilon)$ satisfies $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \le C$ and the function $\hat{f}^\epsilon(x) = \int_{S^\epsilon(x)} f^\epsilon(x,y)\,dy$ satisfies that $\hat{f}^\epsilon \rightharpoonup \hat{f}$, w-$L^2(0,1)$ where $S^\epsilon(x) = \{y \mid (x,y) \in \Omega^\epsilon\}$, that is, the section of the domain Ω^ϵ at the point $x \in (0,1)$.*

Then, if $u_0(x_1)$ is the unique weak solution of the Neumann problem

$$\int_0^1 \Big\{ b(x)\, u_x(x)\, \varphi_x(x) + q(x)\, u(x)\, \varphi(x) \Big\} dx = \int_0^1 \hat{f}(x)\, \varphi\, dx, \quad \forall \varphi \in H^1(0,1)$$

where $q(x)$ is the function given by

$$q(x) = \frac{|Q_0|}{L} + b(x_1) \quad \forall x_1 \in (0,1),$$

we have

$$\|u^\epsilon - u_0\|_{L^2(\Omega^\epsilon)} \overset{\epsilon \to 0}{\longrightarrow} 0.$$

Moreover,

$$u_\epsilon \rightharpoonup u_0 \quad \omega - H^1(\Omega_-).$$

3. An important estimate

In this section we show several basic estimates on the solutions of certain elliptics pde's posed in rectangles of the type

$$Q_\epsilon = \{(x,y) \in \mathbb{R}^2 \mid -\epsilon^\alpha < x < \epsilon^\alpha,\ 0 < y < 1\}$$

with $\alpha > 1$. As a matter of fact, for $u_0(\cdot) \in H^1(-\epsilon^\alpha, \epsilon^\alpha)$, we define the function $u^\epsilon(x,y)$ as the unique solution of

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial x^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial y^2} = 0 \quad \text{in } Q_\epsilon, \\ \quad u(x,0) = u_0(x), \quad \text{on } \Gamma_\epsilon, \\ \quad \dfrac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial Q_\epsilon \setminus \Gamma_\epsilon \end{cases} \tag{3.1}$$

where ν is the outward unit normal to ∂Q_ϵ and

$$\Gamma_\epsilon = \{(x,0) \in \mathbb{R}^2 \mid -\epsilon^\alpha < x < \epsilon^\alpha\}.$$

We have the following,

Lemma 3.1. *With the notation from above, if we denote by $\bar{u}_0$ the average of u_0 in Γ_ϵ, that is*

$$\bar{u}_0 = \frac{1}{2\epsilon^\alpha}\int_{-\epsilon^\alpha}^{\epsilon^\alpha} u_0(x)\,dx \tag{3.2}$$

then there exists a constant C, independent of ϵ and u_0, such that

$$\int_{-\epsilon^\alpha}^{\epsilon^\alpha} |u^\epsilon(x,y) - \bar{u}_0|^2\,dx \le C \exp\left\{-\frac{2y\pi}{\epsilon^{\alpha-1}}\right\} \|u_0\|^2_{L^2(-\epsilon^\alpha,\epsilon^\alpha)} \tag{3.3}$$

$$\int_0^1 \int_{-\epsilon^\alpha}^{\epsilon^\alpha} |u(x,y) - \bar{u}_0|^2\,dxdy \le C\epsilon^{\alpha-1}\|u_0\|^2_{L^2(-\epsilon^\alpha,\epsilon^\alpha)} \tag{3.4}$$

and

$$\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q_\epsilon)} + \frac{1}{\epsilon^2}\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2(Q_\epsilon)} \le C\epsilon^{\alpha-1}\left\|\frac{\partial u_0}{\partial x}\right\|^2_{L^2(-\epsilon^\alpha,\epsilon^\alpha)}. \tag{3.5}$$

Proof. The proof of this result is based in the known fact that the solution of the problem above can be found explicitely and admits a Fourier decomposition of the form

$$u^\epsilon(x,y) = \frac{1}{2\epsilon^\alpha}\int_{-\epsilon^\alpha}^{\epsilon^\alpha} u_0(x)dx + \sum_{k=1}^{\infty}(u_0,\varphi_n^\epsilon)\varphi_n^\epsilon(x)\frac{\cosh(\frac{n\pi(1-y)}{\epsilon^{\alpha-1}})}{\cosh(\frac{n\pi}{\epsilon^{\alpha-1}})} \tag{3.6}$$

where $\varphi_n^\epsilon(x) = \epsilon^{-\alpha/2}\cos(\frac{n\pi x}{\epsilon^\alpha})$, $n = 1, 2, \ldots$, and $(u_0, \varphi_n^\epsilon) = (u_0, \varphi_n^\epsilon)_{L^2(-\epsilon^\alpha,\epsilon^\alpha)}$. □

Remark 3.2. *Observe that in particular, estimate* (3.5) *implies that*

$$\min_{u\in V}\left\{\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(Q_\epsilon)} + \frac{1}{\epsilon^2}\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2(Q_\epsilon)}\right\} \le C\epsilon^{\alpha-1}\left\|\frac{\partial u_0}{\partial x}\right\|^2_{L^2(-\epsilon^\alpha,\epsilon^\alpha)}$$

where $V = \{u \in H^1(Q_\epsilon)\,|\, u = u_0 \text{ in } \Gamma_\epsilon\}$.

4. Thin domains as a region between graphs.

In this section we consider Type I thin domains and provide a proof of Theorem 2.1.

We will start analyzing in detail the structure of the domain Ω^ϵ as a preparation for the proof of our result.

4.1. **The one parameter family G^ϵ.** In this subsection we obtain some properties and a convenient approximation to the parameter family G_ϵ that we will use in the proof of the main result Theorem 2.1.

From $(H2)$ we have that there exists a positive constant G_1 such that

$$0 \le G_\epsilon(x) \le G_1, \quad \forall x \in (0,1), \quad \forall \epsilon \in (0,\epsilon_0). \tag{4.1}$$

Moreover, for each $x \in [0,1]$, we consider the function

$$G_0(x) = \min_{y\in\mathbb{R}} G(x,y) \ge 0. \tag{4.2}$$

We show that $G_0(\cdot)$ is a continuous function in $[0,1]$. Indeed, we will prove that

$$|G_0(x) - G_0(x')| \le \sup_{y\in\mathbb{R}}|G(x,y) - G(x',y)| \quad \forall x, x' \in [0,1]. \tag{4.3}$$

Consequently, the continuity of G_0 follows from the uniform continuity of G in y and inequality (4.3).

Thus, let us prove (4.3). Given x and $x' \in [0,1]$, there exist $y(x)$ and $y(x') \in \mathbb{R}$ such that $G_0(x) = G(x, y(x))$ and $G_0(x') = G(x', y(x'))$. On the one hand, we have

$$G_0(x) - G_0(x') = G_0(x) - G(x,y(x')) + G(x,y(x')) - G(x',y(x')) \le G(x,y(x')) - G(x',y(x')). \tag{4.4}$$

In a completely symmetric fashion we also obtain

$$G_0(x') - G_0(x) \le G(x',y(x)) - G(x,y(x)). \tag{4.5}$$

Consequently, we obtain (4.3) from (4.4) and (4.5).

Now, let us denote by N_ϵ the largest integer such that $N_\epsilon L\epsilon^\alpha < 1$, where L is given in hypothesis (H2). Observe that $N_\epsilon \sim L^{-1}\epsilon^{-\alpha}$. Let

$$G_{n,\epsilon} = \min_{x\in[(n-1)L\epsilon^\alpha, nL\epsilon^\alpha]} G\left(x, \frac{x}{\epsilon^\alpha}\right), \quad n = 1, 2 \ldots, N_\epsilon \tag{4.6}$$

and $\gamma_{n,\epsilon} \in [(n-1)L\epsilon^\alpha, nL\epsilon^\alpha]$ a point where the minimum (4.6) is attained, that is, $G(\gamma_{n,\epsilon}, \frac{\gamma_{n,\epsilon}}{\epsilon^\alpha}) = G_{n,\epsilon}$ where $\gamma_{n,\epsilon}$ does not need to be uniquely defined. By extension, let us denote by $\gamma_{0,\epsilon} = 0$ and $\gamma_{N_\epsilon+1,\epsilon} = 1$.

Note that the set

$$\{\gamma_{0,\epsilon}, \gamma_{1,\epsilon}, ..., \gamma_{N_\epsilon+1,\epsilon}\} \tag{4.7}$$

defines a partition for the unit interval $[0,1]$. Also, we have by definition that the segments

$$\{(\gamma_{n,\epsilon}, x_2) \mid G_{n,\epsilon} < x_2 < G_1\} \cap \Omega^\epsilon = \emptyset,$$

for all $n = 1, 2, \ldots, N_\epsilon$.

Consider also the step function

$$\tilde{G}_0^\epsilon(x) = \begin{cases} G_{1,\epsilon}, & x \in [0, \gamma_{1,\epsilon}] \\ \max\{G_{n,\epsilon}, G_{n+1,\epsilon}\}, & x \in [\gamma_{n,\epsilon}, \gamma_{n+1,\epsilon}], n = 1, 2 \ldots, N_\epsilon - 1 \\ G_{N_\epsilon,\epsilon}, & x \in [\gamma_{N_\epsilon,\epsilon}, 1] \end{cases}. \tag{4.8}$$

Lemma 4.1. *We have*

$$\|G_0 - \tilde{G}_0^\epsilon\|_{L^\infty(0,1)} \to 0 \qquad \text{as } \epsilon \to 0.$$

Proof. It follows from $(H2)$ and (4.3) that, for each $\eta > 0$, there exists $\epsilon_0 > 0$ such that

$$\max\{|G(x,y) - G(x',y)|, |G_0(x) - G_0(x')|\} < \eta \tag{4.9}$$

whenever $|x - x'| < 2\,\epsilon_0^\alpha\, L$ and $y \in \mathbb{R}$. Now, for all $x \in [\gamma_{n,\epsilon}, \gamma_{n+1,\epsilon}]$ we have

$$\tilde{G}_0^\epsilon(x) - G_0(x) = \max\{G_{n,\epsilon}, G_{n+1,\epsilon}\} - G_0(x).$$

Without loss of generality, we may assume $\tilde{G}_0^\epsilon(x) = G_{n,\epsilon}$, that is, $G_{n,\epsilon} \geq G_{n+1,\epsilon}$. Thus,

$$\begin{aligned} \tilde{G}_0^\epsilon(x) - G_0(x) &= G_{n,\epsilon} - G_0(x) \\ &= G(\gamma_{n,\epsilon}, \gamma_{n,\epsilon}/\epsilon^\alpha) - G_0(x) \\ &= G(\gamma_{n,\epsilon}, \gamma_{n,\epsilon}/\epsilon^\alpha) - G_0(\gamma_{n,\epsilon}) + G_0(\gamma_{n,\epsilon}) - G_0(x). \end{aligned} \tag{4.10}$$

It follows from definition of G_0 in (4.2), that

$$G(\gamma_{n,\epsilon}, \gamma_{n,\epsilon}/\epsilon^\alpha) - G_0(\gamma_{n,\epsilon}) \geq 0.$$

Also, since $G(x,\cdot)$ is $l(x)$-periodic with $|l(x)| \leq L$, we have that there exist $y(\gamma_{n,\epsilon}) \in [0, l(\gamma_{n,\epsilon})]$ and $k(\gamma_{n,\epsilon}) \in \mathbb{N}$ with $y(\gamma_{n,\epsilon}) + k(\gamma_{n,\epsilon})\, l(\gamma_{n,\epsilon}) \in [(n-1)\, L\, \epsilon^\alpha, n\, L\, \epsilon^\alpha]$, such that

$$G_0(\gamma_{n,\epsilon}) = G(\gamma_{n,\epsilon}, y(\gamma_{n,\epsilon})) = G(\gamma_{n,\epsilon}, y(\gamma_{n,\epsilon}) + k(\gamma_{n,\epsilon})\, l(\gamma_{n,\epsilon})). \tag{4.11}$$

Consequently, we get from (4.6) and (4.11) that

$$\begin{aligned} G(\gamma_{n,\epsilon}, \gamma_{n,\epsilon}/\epsilon^\alpha) - G_0(\gamma_{n,\epsilon}) &= G(\gamma_{n,\epsilon}, \gamma_{n,\epsilon}/\epsilon^\alpha) - G((y + k\, l)\, \epsilon^\alpha, (y + k\, l)\, \epsilon^\alpha/\epsilon^\alpha) \\ &\quad + G((y + k\, l)\, \epsilon^\alpha, (y + k\, l)\, \epsilon^\alpha/\epsilon^\alpha) - G(\gamma_{n,\epsilon}, (y + k\, l)\, \epsilon^\alpha/\epsilon^\alpha) \\ &\leq G((y + k\, l)\, \epsilon^\alpha, (y + k\, l)\, \epsilon^\alpha/\epsilon^\alpha) - G(\gamma_{n,\epsilon}, (y + k\, l)\, \epsilon^\alpha/\epsilon^\alpha) \end{aligned} \tag{4.12}$$

since

$$G(\gamma_{n,\epsilon}, \gamma_{n,\epsilon}/\epsilon^\alpha) - G((y + k\, l)\, \epsilon^\alpha, (y + k\, l)\, \epsilon^\alpha/\epsilon^\alpha) \leq 0.$$

Therefore, due to (4.10), (4.12) and (4.9), we obtain

$$|\tilde{G}_0^\epsilon(x) - G_0(x)| \quad \leq \quad G((y + k\, l)\, \epsilon^\alpha, (y + k\, l)\, \epsilon^\alpha/\epsilon^\alpha) - G(\gamma_{n,\epsilon}, (y + k\, l)\, \epsilon^\alpha/\epsilon^\alpha) + G_0(\gamma_{n,\epsilon}) - G_0(x) < 2\eta$$

whenever $x \in [\gamma_{n,\epsilon}, \gamma_{n+1,\epsilon}]$.

Then, since $x \in [0,1]$ is arbitrary and $\cup_{n=1}^{N_\epsilon} [\gamma_{n,\epsilon}, \gamma_{n+1,\epsilon}] = [0,1]$, we conclude the proof. $\square$

The following result will also be needed.

Lemma 4.2. *We have the following*

$$G_\epsilon(\cdot) \stackrel{\epsilon \to 0}{\rightharpoonup} \frac{1}{l(\cdot)} \int_0^{l(\cdot)} G(\cdot, s)\, ds \qquad w^* - L^\infty(0,1). \tag{4.13}$$

Proof. We have to prove

$$\int_0^1 \left\{ G_\epsilon(x) - \frac{1}{l(x)} \int_0^{l(x)} G(x,s)\, ds \right\} \varphi(x)\, dx \to 0 \qquad \text{as } \epsilon \to 0 \tag{4.14}$$

for all $\varphi \in L^1(0,1)$.

Since the set of step function is dense in $L^1(0,1)$, and any step function is a linear combination of characteristic functions, we only need to show (4.14) for characteristic functions. Then, for $0 \le e < f \le 1$ we consider the following characteristic function

$$\varphi(x) = \left\{ \begin{array}{ll} 1 & x \in (e,f) \\ 0 & x \notin (e,f) \end{array} \right. .$$

So, we have to estimate the integral

$$I_{e,f} = \int_e^f \left\{ G_\epsilon(x) - \frac{1}{l(x)} \int_0^{l(x)} G(x,s)\, ds \right\} dx$$

as $\epsilon > 0$ goes to zero.

For this, let $\eta > 0$ be a small number and let $\{e = x_0, x_1, ..., x_n = f\}$ be a partition for the interval (e,f), and $\hat{x}_i$ be a fixed point in the interval $J_i = [x_{i-1}, x_i]$, $i = 1, ..., n$, such that

$$\sup_i \sup_{x \in J_i,\, y \in \mathbb{R}} |G(x,y) - G(\hat{x}_i, y)| < \eta.$$

Observe that we can write

$$I_{e,f} = \sum_{i=1}^{5} I_{e,f}^i$$

where

$$\begin{aligned}
I_{e,f}^1 &= \sum_{i=1}^n \int_{J_i} \{G(x, x/\epsilon^\alpha) - G(\hat{x}_i, x/\epsilon^\alpha)\}\, dx \\
I_{e,f}^2 &= \sum_{i=1}^n \int_{J_i} \left\{ G(\hat{x}_i, x/\epsilon^\alpha) - \frac{1}{l(\hat{x}_i)} \int_0^{l(\hat{x}_i)} G(\hat{x}_i, s)\, ds \right\} dx \\
I_{e,f}^3 &= \sum_{i=1}^n \int_{J_i} \left\{ \frac{1}{l(\hat{x}_i)} \int_0^{l(\hat{x}_i)} G(\hat{x}_i, s)\, ds - \frac{1}{l(\hat{x}_i)} \int_0^{l(\hat{x}_i)} G(x, s)\, ds \right\} dx \\
I_{e,f}^4 &= \sum_{i=1}^n \int_{J_i} \left\{ \frac{1}{l(\hat{x}_i)} \int_0^{l(\hat{x}_i)} G(x, s)\, ds - \frac{1}{l(\hat{x}_i)} \int_0^{l(x)} G(x, s)\, ds \right\} dx \\
I_{e,f}^5 &= \sum_{i=1}^n \int_{J_i} \left\{ \frac{1}{l(\hat{x}_i)} \int_0^{l(x)} G(x, s)\, ds - \frac{1}{l(x)} \int_0^{l(x)} G(x, s)\, ds \right\} dx.
\end{aligned}$$

It is easy to estimate the integrals $I_{e,f}^1$, $I_{e,f}^3$, $I_{e,f}^4$ and $I_{e,f}^5$ to obtain

$$\begin{aligned}
|I_{e,f}^1| &\le \eta\, (f-e) \\
|I_{e,f}^3| &\le \eta\, (f-e) \\
|I_{e,f}^4| &\le G_1\, \|\hat{l}^\eta - l\|_{L^\infty(0,1)}\, (f-e) \\
|I_{e,f}^5| &\le G_1\, \left(L/{L'}^2 \right)\, \|\hat{l}^\eta - l\|_{L^\infty(0,1)}\, (f-e)
\end{aligned} \tag{4.15}$$

where G_1, L and L' are the positive constants given by hypothesis $(H2)$, and the function $\hat{l}^\eta$ is the step function defined for each $\eta > 0$ by

$$\hat{l}^\eta(x) = l(x_i) \qquad \text{as } x_i \in J_i.$$

Since the inequalities (4.15) do not depend on $\epsilon > 0$, and $\|\hat{l}^\eta - l\|_{L^\infty(0,1)} \to 0$ as $\eta \to 0$ uniformly in ϵ, we have that $I^1_{e,f}$, $I^3_{e,f}$, $I^4_{e,f}$ and $I^5_{e,f}$ go to zero as $\eta \to 0$ uniformly in $\epsilon > 0$.

Hence, to conclude the proof, we just evaluate the integral $I^2_{e,f}$. But this is a straightforward application of the Average Theorem since $\hat{x}_i$ is a fixed point in J_i, and $G(\hat{x}_i, \cdot)$ is a $l(\hat{x}_i)$-periodic function. Indeed,

$$I^2_{e,f} = \sum_{i=1}^n \int_{J_i} \left\{ G(\hat{x}_i, x/\epsilon^\alpha) - \frac{1}{l(\hat{x}_i)} \int_0^{l(\hat{x}_i)} G(\hat{x}_i, s)\, ds \right\} dx \to 0 \text{ as } \epsilon \to 0.$$

□

4.2. **Proof of Theorem 2.1.** Here, we give a proof of the main result, Theorem 2.1.

Proof. The variational formulation of (2.2) is: find $u^\epsilon \in H^1(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \varphi dx_1 dx_2, \quad \forall \varphi \in H^1(\Omega^\epsilon). \tag{4.16}$$

Taking $\varphi = u^\epsilon$ in expression (4.16) and using that $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \le C$, we easily obtain the a priori bounds

$$\|u^\epsilon\|_{L^2(\Omega^\epsilon)}, \Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|_{L^2(\Omega^\epsilon)} \text{ and } \frac{1}{\epsilon}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega^\epsilon)} \le C. \tag{4.17}$$

In particular, we have

$$\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega^\epsilon)} \le \epsilon\, C \to 0 \text{ as } \epsilon \to 0.$$

Let us observe that domain Ω^ϵ consists of two main parts. One of them is a highly oscillating domain Ω^ϵ_+ and the other one is a non oscillating domain Ω^ϵ_-. To define these domains, we use the partition for the unit interval $(0,1)$ given by (4.7) and the step function $\tilde{G}^\epsilon_0$ defined in (4.8). So, we consider the following open sets

$$\begin{aligned} \Omega^\epsilon_- &= \{(x_1,x_2) \in \mathbb{R}^2 \,|\, x_1 \in (0,1),\, -b(x_1) < x_2 < \tilde{G}^\epsilon_0(x_1)\} \\ \Omega^\epsilon_+ &= \{(x_1,x_2) \in \mathbb{R}^2 \,|\, x_1 \in (0,1),\, \tilde{G}^\epsilon_0(x_1) < x_2 < G_\epsilon(x_1)\}. \end{aligned} \tag{4.18}$$

Notice that

$$\Omega^\epsilon = \text{Int}\left(\overline{\Omega^\epsilon_+ \cup \Omega^\epsilon_-}\right).$$

Observe also that with the function G_0 defined by (4.2) we may also consider the set

$$\Omega_0 = \{(x_1,x_2) \in \mathbb{R}^2 \,|\, x_1 \in (0,1),\, -b(x_1) < x_2 < G_0(x_1)\}.$$

which satisfies $\Omega_0 \subset \Omega^\epsilon$.

We want to pass to the limit in the variational formulation (4.16) for certain appropriately chosen test functions. In order to accomplish this, we rewrite it as follows

$$\begin{aligned} &\int_{\Omega^\epsilon_+} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} \Big\} dx_1 dx_2 + \int_{\Omega^\epsilon_-} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} \Big\} dx_1 dx_2 \\ &\quad + \int_{\Omega^\epsilon} u^\epsilon \varphi \, dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \varphi \, dx_1 dx_2, \, \forall \varphi \in H^1(\Omega^\epsilon). \end{aligned} \tag{4.19}$$

Now, we pass to the limit in the different functions that form the integrands of (4.19).

(a). Limit of u^ϵ in L^2.

It follows from (4.17) that $u^\epsilon|_{\Omega_0} \in H^1(\Omega_0)$ and satisfies for all $\epsilon > 0$

$$\|u^\epsilon\|_{L^2(\Omega_0)}, \Big\|\frac{\partial u^\epsilon}{\partial x_1}\Big\|_{L^2(\Omega_0)} \text{ and } \frac{1}{\epsilon}\Big\|\frac{\partial u^\epsilon}{\partial x_2}\Big\|_{L^2(\Omega_0)} \le C.$$

Then, we can extract a subsequence of $\{u^\epsilon|_{\Omega_0}\} \subset H^1(\Omega_0)$, denoted again by u^ϵ, such that

$$u^\epsilon \to u_0 \quad \begin{array}{l} u^\epsilon \rightharpoonup u_0 \quad w - H^1(\Omega_0) \\ s - H^s(\Omega_0) \text{ for all } s \in [0,1) \text{ and} \\ \dfrac{\partial u^\epsilon}{\partial x_2} \to 0 \quad s - L^2(\Omega_0) \end{array} \tag{4.20}$$

as $\epsilon \to 0$ for some $u_0 \in H^1(\Omega_0)$.

A consequence of the limits (4.20) is that $u_0(x_1, x_2)$ does not depend on the variable x_2. More precisely,

$$\frac{\partial u_0}{\partial x_2}(x_1, x_2) = 0 \text{ a.e. } \Omega_0. \tag{4.21}$$

Also, due to (4.20), we have that the restriction of u^ϵ to the coordinate axis x_1 converges to u_0. That is,

$$u^\epsilon|_\Gamma \to u_0 \quad s - H^s(\Gamma)$$

for all $s \in [0, 1/2)$ where $\Gamma = \{(x_1, 0) \in \mathbb{R}^2 \,|\, x_1 \in (0,1)\}$. Consequently, we obtain

$$\|u^\epsilon - u_0\|_{L^2(\Gamma)} \to 0 \text{ as } \epsilon \to 0. \tag{4.22}$$

Now, we can see that (4.22) implies the L^2-convergence of u^ϵ to u_0, that is

$$\|u^\epsilon - u_0\|_{L^2(\Omega^\epsilon)} \to 0 \text{ as } \epsilon \to 0. \tag{4.23}$$

In fact, it follows from (4.22) that

$$\begin{aligned} \|u^\epsilon(x_1, 0) - u_0(x_1)\|^2_{L^2(\Omega^\epsilon)} &= \int_0^1 \int_{-b(x_1)}^{G_\epsilon(x_1)} |u^\epsilon(x_1, 0) - u_0(x_1)|^2 \, dx_2 dx_1 \\ &\leq C(G, b) \, \|u^\epsilon - u_0\|_{L^2(\Gamma)} \\ &\to 0 \text{ as } \epsilon \to 0 \end{aligned}$$

where $C(G, b)$ is a generic constant that depends on the functions G and b. Also,

$$u^\epsilon(x_1, x_2) - u^\epsilon(x_1, 0) = \int_0^{x_2} \frac{\partial u^\epsilon}{\partial x_2}(x_1, s) \, ds$$

and with Hölder inequality,

$$|u^\epsilon(x_1, x_2) - u^\epsilon(x_1, 0)|^2 \leq \left(\int_0^{x_2} \left| \frac{\partial u^\epsilon}{\partial x_2}(x_1, s) \right|^2 ds \right) |x_2|.$$

Hence, integrating in Ω^ϵ and using (4.17) to get

$$\|u^\epsilon(x_1, x_2) - u^\epsilon(x_1, 0)\|^2_{L^2(\Omega^\epsilon)} = \int_0^1 \int_{-b(x_1)}^{G_\epsilon(x_1)} |u^\epsilon(x_1, x_2) - u^\epsilon(x_1, 0)|^2 \, dx_1 dx_2$$
$$\leq \int_0^1 \int_{-b(x_1)}^{G_\epsilon(x_1)} \left(\int_0^{x_2} \left| \frac{\partial u^\epsilon}{\partial x_2}(x_1, s) \right|^2 ds \right) |x_2| \, dx_2 dx_1 \leq C(G, b) \left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|^2_{L^2(\Omega^\epsilon)} \leq \epsilon \, \hat{C}(G, b) \to 0 \text{ as } \epsilon \to 0.$$

Therefore,

$$\begin{aligned} \|u^\epsilon - u_0\|_{L^2(\Omega^\epsilon)} &\leq \|u^\epsilon(x_1, x_2) - u^\epsilon(x_1, 0)\|_{L^2(\Omega^\epsilon)} + \|u^\epsilon(x_1, 0) - u_0(x_1)\|_{L^2(\Omega^\epsilon)} \\ &\to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

(b). Limit of f^ϵ.

Since $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \leq C$ independent of ϵ, we have the function $\hat{f}^\epsilon$ defined by

$$\hat{f}^\epsilon(x_1) \equiv \int_{-b(x_1)}^{G_\epsilon(x_1)} f^\epsilon(x_1, x_2) dx_2 \tag{4.24}$$

belongs to $L^2(0,1)$ and satisfies $\|\hat{f}^\epsilon\|_{L^2(0,1)} \leq C(G,b)$ for some constant $C(G,b)$ independent of ϵ. Hence, via subsequences, we have the existence of a function $\hat{f} = \hat{f}(x_1) \in L^2(0,1)$ such that

$$\hat{f}^\epsilon \rightharpoonup \hat{f} \qquad w - L^2(0,1). \tag{4.25}$$

Remark 4.3. *Observe that in the case where* $f^\epsilon(x_1, x_2) = f(x_1)$ *then*

$$\hat{f}^\epsilon(x_1) = (G(x_1, x_1/\epsilon^\alpha) + b(x_1))\ f(x_1) \rightharpoonup p(x_1)\, f(x_1) \qquad w^* - L^\infty(0,1)$$

where the function p *is given by*

$$p(x) = \frac{1}{l(x)} \int_0^{l(x)} G(x,s)\, ds + b(x), \tag{4.26}$$

which is the weak $*$-$L^\infty(0,1)$ *limit of* $G_\epsilon(x)$ *obtained in (4.13). Consequently, we have that*

$$\hat{f}(x) = p(x)\, f(x) \quad x \in (0,1).$$

(c). Test functions.

Here, we define suitable test functions that will allow us to pass the limit in the variational formulation (4.19). For this, we use the definition of the open sets Ω_-^ϵ and Ω_+^ϵ given in (4.18).

For each $\phi \in H^1(0,1)$ and $\epsilon > 0$, we define the following test functions in $H^1(\Omega^\epsilon)$

$$\varphi^\epsilon(x_1,x_2) = \begin{cases} X_n^\epsilon(x_1,x_2), & (x_1,x_2) \in \Omega_+^\epsilon \cap Q_n^\epsilon, \quad n = 1, 2, \dots \\ \phi(x_1), & (x_1,x_2) \in \Omega_-^\epsilon \end{cases} \tag{4.27}$$

where Q_n^ϵ is the rectangle (see Figure 4)

$$Q_n^\epsilon = \{(x_1,x_2) \,|\, \gamma_{n,\epsilon} < x_1 < \gamma_{n+1,\epsilon},\, \tilde{G}_0^\epsilon(x_1) < x_2 < G_1\}$$

and the function X_n^ϵ is the solution of the problem

$$\begin{cases} -\dfrac{\partial^2 X^\epsilon}{\partial x_1^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 X^\epsilon}{\partial x_2^2} = 0, \quad \text{ in } Q_n^\epsilon \\ \qquad \dfrac{\partial X^\epsilon}{\partial N^\epsilon} = 0, \quad \text{ on } \partial Q_n^\epsilon \backslash \Gamma_n^\epsilon \\ \quad X^\epsilon(x_1,x_2) = \phi(x_1), \quad \text{ on } \Gamma_n^\epsilon \end{cases} \tag{4.28}$$

where Γ_n^ϵ is the base of the rectangle, that is,

$$\Gamma_n^\epsilon = \{(x_1, \tilde{G}_0^\epsilon(x_1)) : \gamma_{n,\epsilon} \leq x_1 \leq \gamma_{n+1,\epsilon}\}.$$

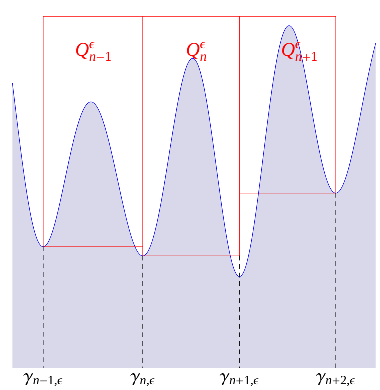

FIGURE 4. Rectangle Q_n^ϵ.

It follows from estimate (3.5) that

$$\left\|\frac{\partial^2 X^\epsilon}{\partial x_1^2}\right\|^2_{L^2(Q_n^\epsilon)} + \frac{1}{\epsilon^2}\left\|\frac{\partial^2 X^\epsilon}{\partial x_2^2}\right\|^2_{L^2(Q_n^\epsilon)} \le C\epsilon^{\alpha-1}\|\phi'\|^2_{L^2(\gamma_{n,\epsilon},\gamma_{n+1,\epsilon})}. \tag{4.29}$$

Moreover, if we call $Q^\epsilon = \cup_{i=1}^{N_\epsilon} Q_n^\epsilon$, we get $\Omega_+^\epsilon = Q^\epsilon \cap \Omega^\epsilon$, and we can define the function X^ϵ in Ω_+^ϵ by

$$X^\epsilon(x_1,x_2) = X_n^\epsilon(x_1,x_2) \text{ as } (x_1,x_2) \in Q_n^\epsilon \cap \Omega^\epsilon.$$

Hence, we have by (4.29) that $X^\epsilon \in H^1(\Omega_+^\epsilon)$ and satisfies the following inequality

$$\begin{aligned}\left\|\frac{\partial^2 X^\epsilon}{\partial x_1^2}\right\|^2_{L^2(\Omega_+^\epsilon)} + \frac{1}{\epsilon^2}\left\|\frac{\partial^2 X^\epsilon}{\partial x_2^2}\right\|^2_{L^2(\Omega_+^\epsilon)} &\le \sum_{i=1}^{N_\epsilon}\left(\left\|\frac{\partial^2 X^\epsilon}{\partial x_1^2}\right\|^2_{L^2(Q_n^\epsilon)} + \frac{1}{\epsilon^2}\left\|\frac{\partial^2 X^\epsilon}{\partial x_2^2}\right\|^2_{L^2(Q_n^\epsilon)}\right) \\ &\le \sum_{i=1}^{N_\epsilon} C\,\epsilon^{\alpha-1}\,\|\phi'\|^2_{L^2(\gamma_{\epsilon,n},\gamma_{n+1,\epsilon})} \le C\,\epsilon^{\alpha-1}\,\|\phi'\|^2_{L^2(0,1)}.\end{aligned} \tag{4.30}$$

Furthermore, we can show that

$$\|\varphi^\epsilon - \phi\|_{L^2(\Omega^\epsilon)} \to 0 \text{ as } \epsilon \to 0. \tag{4.31}$$

We can argue as in (4.23). Indeed, since

$$\varphi^\epsilon(x_1,x_2) - \phi(x_1) = \varphi^\epsilon(x_1,x_2) - \varphi^\epsilon(x_1,0) = \int_0^{x_2}\frac{\partial\varphi^\epsilon}{\partial x_2}(x_1,s)\,ds,$$

we have by (4.27) and (4.30) that

$$\|\varphi^\epsilon - \phi\|^2_{L^2(\Omega^\epsilon)} \le C(G,b)\left\|\frac{\partial\varphi^\epsilon}{\partial x_2}\right\|^2_{L^2(\Omega^\epsilon)} \le C(G,b)\left\|\frac{\partial X^\epsilon}{\partial x_2}\right\|^2_{L^2(Q^\epsilon\cap\Omega^\epsilon)} \le \hat{C}(G,b)\,\epsilon^{1+\alpha}\,\|\phi'\|^2_{L^2(0,1)} \to 0 \text{ as } \epsilon \to 0.$$

(d). Passing to the limit in the weak formulation.

Now, we pass to the limit in the variational formulation of the problem using the test functions φ^ϵ defined above. For this, we analyze the different functions that form the integrands in (4.19).

- First integrand:

$$\int_{\Omega_+^\epsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial\varphi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial\varphi^\epsilon}{\partial x_2}\Big\}dx_1dx_2 \to 0 \text{ as } \epsilon \to 0. \tag{4.32}$$

Indeed, it follows from (4.30) and $\alpha > 0$ that

$$\begin{aligned}&\int_{\Omega_+^\epsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial\varphi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial\varphi^\epsilon}{\partial x_2}\Big\}dx_1dx_2 = \int_{\Omega_+^\epsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial X^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial X^\epsilon}{\partial x_2}\Big\}dx_1dx_2 \\ &\le \left(\int_{\Omega_u^\epsilon}\Big\{\left(\frac{\partial u^\epsilon}{\partial x_1}\right)^2 + \frac{1}{\epsilon^2}\left(\frac{\partial u^\epsilon}{\partial x_2}\right)^2\Big\}dx_1dx_2\right)^{1/2}\left(\int_{\Omega_+^\epsilon}\Big\{\left(\frac{\partial X^\epsilon}{\partial x_1}\right)^2 + \frac{1}{\epsilon^2}\left(\frac{\partial X^\epsilon}{\partial x_2}\right)^2\Big\}dx_1dx_2\right)^{1/2} \\ &\le C\,\epsilon^{(\alpha-1)/2}\,\|u^\epsilon\|_{H^1(\Omega^\epsilon)}\,\|\phi'\|_{L^2(0,1)} \to 0 \text{ as } \epsilon \to 0.\end{aligned} \tag{4.33}$$

- Second integrand:

$$\int_{\Omega_-^\epsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial\varphi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial\varphi^\epsilon}{\partial x_2}\Big\}dx_1dx_2 \to \int_0^1 (G_0(x_1) + b(x_1))\;u_0'(x_1)\,\phi'(x_1)\,dx_1 \text{ as } \epsilon \to 0. \tag{4.34}$$

To prove this, observe that using (4.27), we obtain

$$\frac{\partial\varphi^\epsilon}{\partial x_1}\Big|_{\Omega_-^\epsilon} = \frac{\partial\phi}{\partial x_1} = \phi' \quad \text{ and } \quad \frac{\partial\varphi^\epsilon}{\partial x_2}\Big|_{\Omega_-^\epsilon} = \frac{\partial\phi}{\partial x_2} = 0$$

for all $\epsilon > 0$. Hence, we have that

$$\int_{\Omega_-^\epsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial\varphi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial\varphi^\epsilon}{\partial x_2}\Big\}dx_1dx_2 = \int_{\Omega_-^\epsilon}\frac{\partial u^\epsilon}{\partial x_1}(x_1,x_2)\,\phi'(x_1)\,dx_1dx_2$$

$$= \int_{\Omega_0} \frac{\partial u^\epsilon}{\partial x_1}(x_1,x_2)\,\phi'(x_1)\,dx_1dx_2 - \int_{\Omega_0\setminus\Omega^\epsilon_-} \frac{\partial u^\epsilon}{\partial x_1}(x_1,x_2)\,\phi'(x_1)\,dx_1dx_2$$
$$+ \int_{\Omega^\epsilon_-\setminus\Omega_0} \frac{\partial u^\epsilon}{\partial x_1}(x_1,x_2)\,\phi'(x_1)\,dx_1dx_2. \tag{4.35}$$

Due to (4.20), we can pass to the limit as $\epsilon \to 0$ in the first integral of the right side of (4.35) to obtain

$$\int_{\Omega_0} \frac{\partial u^\epsilon}{\partial x_1}(x_1,x_2)\,\phi'(x_1)\,dx_1dx_2 \quad \to \quad \int_{\Omega_0} u_0'(x_1)\,\phi'(x_1)\,dx_1dx_2.$$

Also, we have that

$$\begin{aligned}\int_{\Omega_0} u_0'(x_1)\,\phi'(x_1)\,dx_1dx_2 &= \int_0^1 u_0'(x_1)\,\phi'(x_1)\left(\int_{-b(x_1)}^{G_0(x_1)} dx_2\right)dx_1\\ &= \int_0^1 (G_0(x_1)+b(x_1))\; u_0'(x_1)\,\phi'(x_1)\,dx_1.\end{aligned} \tag{4.36}$$

Now, we will get (4.34) if we prove that the remaining integrals of (4.35) tend to zero as $\epsilon \to 0$. We evaluate one of them, the other calculus is similar.

From (4.17), (4.18) and Remark 4.1, we have that

$$\begin{aligned}\int_{\Omega^\epsilon_-\setminus\Omega_0} \frac{\partial u^\epsilon}{\partial x_1}(x_1,x_2)\,\phi'(x_1)\,dx_1dx_2 &\le \left\|\frac{\partial u^\epsilon}{\partial x_1}\right\|_{L^2(\Omega^\epsilon)} \|\phi'\|_{L^2(\Omega^\epsilon_-\setminus\Omega_0)}\\ &\le C\left\{\int_0^1 \phi'(x_1)^2\left|G_0(x_1)-\tilde G_0^\epsilon(x_1)\right|\,dx_1\right\}^{1/2}\\ &\le C\,\|\phi'\|_{L^2(0,1)}\,\|G_0-\tilde G_0^\epsilon\|^{1/2}_{L^\infty(0,1)}\\ &\to 0 \text{ as } \epsilon\to 0.\end{aligned} \tag{4.37}$$

Therefore, we obtain (4.34) from (4.36) and (4.37).

- Third integrand: if $p(x)$ is defined in (4.26) then,

$$\int_{\Omega^\epsilon} u^\epsilon\,\varphi^\epsilon\,dx_1dx_2 \to \int_0^1 p(x_1)\,u_0(x_1)\,\phi(x_1)\,dx_1 \text{ as } \epsilon\to 0 \tag{4.38}$$

To prove (4.38), observe that

$$\int_{\Omega^\epsilon} u^\epsilon\,\varphi^\epsilon\,dx_1dx_2 = \int_{\Omega^\epsilon} (u^\epsilon-u_0)\;\varphi^\epsilon\,dx_1dx_2 + \int_{\Omega^\epsilon} u_0\;(\varphi^\epsilon-\phi)\;dx_1dx_2 + \int_{\Omega^\epsilon} u_0\,\phi\,dx_1dx_2.$$

From (4.23) and (4.31), we have

$$\int_{\Omega^\epsilon} (u^\epsilon-u_0)\;\varphi^\epsilon\,dx_1dx_2 \to 0 \quad \text{and} \int_{\Omega^\epsilon} u_0\;(\varphi^\epsilon-\phi)\;dx_1dx_2 \to 0, \text{ as } \epsilon\to 0$$

Hence, since

$$\int_{\Omega^\epsilon} u_0(x_1)\,\phi(x_1)\,dx_1dx_2 = \int_0^1 u_0(x_1)\,\phi(x_1)\;(G_\epsilon(x_1)+b(x_1))\;dx_1,$$

we get (4.38) from (4.13).

- Fourth integrand:

$$\int_{\Omega^\epsilon} f^\epsilon\,\varphi^\epsilon\,dx_1dx_2 \to \int_0^1 \hat f(x_1)\,\phi(x_1)\,dx_1 \text{ as } \epsilon\to 0. \tag{4.39}$$

For this, let $\hat{f} \in L^2(0,1)$ be the function defined in (4.24). Since

$$\int_{\Omega^\epsilon} f^\epsilon \, \varphi^\epsilon \, dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \, (\varphi^\epsilon - \phi) \, dx_1 dx_2 + \int_{\Omega^\epsilon} f^\epsilon \, \phi \, dx_1 dx_2$$

and

$$\int_{\Omega^\epsilon} f^\epsilon \, \phi \, dx_1 dx_2 = \int_0^1 \left(\int_{-b(x_1)}^{G_\epsilon(x_1)} f^\epsilon(x_1, x_2) \, dx_2 \right) \phi(x_1) \, dx_1 = \int_0^1 \hat{f}^\epsilon(x_1) \, \phi(x_1) \, dx_1,$$

we get (4.39) from (2.3), (4.25) and (4.31).

Therefore, ifrom (4.32), (4.34), (4.38) and (4.39) we obtain the following limit variational formulation

$$\int_0^1 \{(G_0(x_1) + b(x_1)) \; u_0'(x_1) \, \phi'(x_1) + p(x_1) \, u_0(x_1) \, \phi(x_1)\} \, dx_1 = \int_0^1 \hat{f}(x_1) \, \phi(x_1) \, dx_1 \quad \forall \phi \in H^1(0,1). \tag{4.40}$$

Since this problem is well posed, we obtain that the sequence $\{u^\epsilon\}_{\epsilon > 0}$ is convergent and converges to the unique solution u_0 of (4.40). Thus we conclude the proof of Theorem 2.1. □

5. Comb-like thin domains

We consider now, Type II thin domains as described in Section 2 and provide a proof of Theorem 2.5.

Proof. We will proceed as in the previous section to show this result. We will choose appropriate test functions to pass to the limit in the variational formulation of problem (2.8) that we rewrite it here as: find $u^\epsilon \in H^1(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon_+} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} \Big\} dx_1 dx_2 + \int_{\Omega_-} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} \Big\} dx_1 dx_2$$
$$+ \int_{\Omega^\epsilon} u^\epsilon \varphi \, dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \varphi \, dx_1 dx_2, \, \forall \varphi \in H^1(\Omega^\epsilon). \tag{5.1}$$

Again, as in the previous case, taking $\varphi = u^\epsilon$ in expression (5.1) and using that $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \le C$, we easily obtain the apriori bounds

$$\|u^\epsilon\|_{L^2(\Omega^\epsilon)}, \Big\| \frac{\partial u^\epsilon}{\partial x_1} \Big\|_{L^2(\Omega^\epsilon)} \text{ and } \frac{1}{\epsilon} \Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|_{L^2(\Omega^\epsilon)} \le C. \tag{5.2}$$

In particular, we have

$$\Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|_{L^2(\Omega^\epsilon)} \le \epsilon \, C \to 0 \text{ as } \epsilon \to 0.$$

We extract a subsequence of $\{u^\epsilon|_{\Omega_-}\} \subset H^1(\Omega_-)$, denoted again by u^ϵ, such that

$$\begin{aligned} u^\epsilon &\rightharpoonup u_0 \quad w - H^1(\Omega_-) \\ u^\epsilon \to u_0 \quad & s - H^s(\Omega_-) \text{ for all } s \in [0,1) \text{ and} \\ \frac{\partial u^\epsilon}{\partial x_2} &\to 0 \quad s - L^2(\Omega_-) \end{aligned} \tag{5.3}$$

as $\epsilon \to 0$ for some $u_0 \in H^1(\Omega_-)$.

As in (4.21), it follows from (5.3) that $u_0(x_1, x_2)$ does not depend on the variable x_2 and belongs to $H^1(0,1)$. Indeed, we can show that

$$\frac{\partial u_0}{\partial x_2}(x_1, x_2) = 0 \text{ a.e. } \Omega_-.$$

(a). Limit of u^ϵ in $L^2(\Omega^\epsilon)$.

First, we obtain the L^2-convergence of u^ϵ to u_0. More precisely, we show

$$\|u^\epsilon - u_0\|_{L^2(\Omega^\epsilon)} \to 0 \text{ as } \epsilon \to 0. \tag{5.4}$$

For this, we assume without loss of generality that

$$\Omega_+^\epsilon \subset \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1), \quad 0 < x_2 < b(x_1)\}$$

and we define by 'symmetrization' the following function $\hat{u}^\epsilon$ in Ω_+^ϵ by

$$\hat{u}^\epsilon(x_1,x_2) = \begin{cases} u^\epsilon(x_1,-x_2), & (x_1,x_2) \in \Omega_+^\epsilon \\ u^\epsilon(x_1,x_2), & (x_1,x_2) \in \Omega_-. \end{cases} \tag{5.5}$$

Consequently, it follows from (5.3) that

$$\|\hat{u}^\epsilon - u_0\|_{L^2(\Omega^\epsilon)} \to 0 \text{ as } \epsilon \to 0,$$

and from (5.2), we have

$$\|\hat{u}^\epsilon\|_{L^2(\Omega^\epsilon)}, \left\|\frac{\partial \hat{u}^\epsilon}{\partial x_1}\right\|_{L^2(\Omega^\epsilon)} \text{ and } \frac{1}{\epsilon}\left\|\frac{\partial \hat{u}^\epsilon}{\partial x_2}\right\|_{L^2(\Omega^\epsilon)} \le C. \tag{5.6}$$

Let us denote by $w^\epsilon = u^\epsilon - \hat{u}^\epsilon$ in Ω^ϵ. It is easy to see that $w^\epsilon \equiv 0$ in Ω_- and w^ϵ satisfies

$$\begin{gathered} \|w^\epsilon\|_{H^1(\Omega_+^\epsilon)} = \|u^\epsilon - \hat{u}^\epsilon\|_{H^1(\Omega_+^\epsilon)} \le C_1 \\ \left\|\frac{\partial w^\epsilon}{\partial x_1}\right\|^2_{L^2(\Omega_+^\epsilon)} + \frac{1}{\epsilon^2}\left\|\frac{\partial w^\epsilon}{\partial x_2}\right\|^2_{L^2(\Omega_+^\epsilon)} + \|w^\epsilon\|^2_{L^2(\Omega_+^\epsilon)} \le C_2 \end{gathered} \tag{5.7}$$

Now let us show that $\|w^\epsilon\|_{L^2(\Omega^\epsilon)} \to 0$ as $\epsilon \to 0$, that is, $\|w^\epsilon\|_{L^2(\Omega_+^\epsilon)} \to 0$ as $\epsilon \to 0$. Suppose this is not true and assume that $\|w^\epsilon\|^2_{L^2(\Omega_+^\epsilon)} \ge c_0 > 0$ at least for a subsequence $\epsilon \to 0$. Then we have that

$$J(w^\epsilon) = \frac{\left\|\frac{\partial w^\epsilon}{\partial x_1}\right\|^2_{L^2(\Omega_+^\epsilon)} + \frac{1}{\epsilon^2}\left\|\frac{\partial w^\epsilon}{\partial x_2}\right\|^2_{L^2(\Omega_+^\epsilon)} + \|w^\epsilon\|^2_{L^2(\Omega_+^\epsilon)}}{\|w^\epsilon\|^2_{L^2(\Omega_+^\epsilon)}} \le \frac{C_2}{c_0} = C.$$

This implies that the first eigenvalue of the problem

$$\begin{cases} -\dfrac{\partial^2 v^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 v^\epsilon}{\partial {x_2}^2} + v^\epsilon = \lambda_\epsilon\, v^\epsilon \quad \text{in } \Omega_+^\epsilon \\ \dfrac{\partial v^\epsilon}{\partial x_1}\nu_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial v^\epsilon}{\partial x_2}\nu_2^\epsilon = 0 \quad \text{on } \partial\Omega_+^\epsilon \backslash \Gamma \\ v^\epsilon(x_1,0) = 0 \quad \text{on } \Gamma \end{cases} \tag{5.8}$$

satisfies $\lambda_\epsilon(\Omega_+^\epsilon) \le C$, since J is the associated Raleigh quotient and $\Gamma \subset \partial\Omega_+^\epsilon$ is a nonempty open subset.

But observe that $\Omega_+^\epsilon = \cup_{n=1}^{N_\epsilon} \Omega_{n,+}^\epsilon$ where all $\Omega_{n,+}^\epsilon$ are disjoint and identical, except for translations. Then, we can conclude $\lambda_\epsilon(\Omega_+^\epsilon) = \lambda_\epsilon(\Omega_{n,+}^\epsilon)$ for all n.

Performing in $\Omega_{n,+}^\epsilon$ the change of variables that transforms it into the fixed domain Q_0, that is, $(x_1,x_2) \to (x_1/\epsilon^\alpha - nL, x_2)$, we will have that $\lambda_\epsilon(\Omega_{n,+}^\epsilon)$ is the first eigenvalue of the problem

$$\begin{cases} -\dfrac{1}{\epsilon^{2\alpha}}\dfrac{\partial^2 v^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 v^\epsilon}{\partial {x_2}^2} + v^\epsilon = \lambda_\epsilon\, v^\epsilon \quad \text{in } Q_0 \\ \dfrac{1}{\epsilon^{2\alpha}}\dfrac{\partial v^\epsilon}{\partial x_1}\nu_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial v^\epsilon}{\partial x_2}\nu_2^\epsilon = 0 \quad \text{on } \partial Q_0 \backslash \Gamma_0 \\ v^\epsilon(x_1,0) = 0 \quad \text{on } \Gamma_0 \end{cases} \tag{5.9}$$

and therefore,

$$\lambda_\epsilon(\Omega_+^\epsilon) = \min\left\{ \frac{\frac{1}{\epsilon^{2\alpha}}\int_{Q_0}\left|\frac{\partial^2 v^\epsilon}{\partial {x_1}^2}\right|^2 dx_1dx_2 + \frac{1}{\epsilon^2}\int_{Q_0}\left|\frac{\partial^2 v^\epsilon}{\partial {x_2}^2}\right|^2 dx_1dx_2}{\int_{Q_0}|v^\epsilon|^2\, dx_1dx_2}\ ;\ v^\epsilon \in H^1(Q_0),\ v^\epsilon|_{\Gamma_0} = 0 \right\} \le C.$$

But this is impossible since $\lambda_\epsilon(\Omega_+^\epsilon) \geq \frac{1}{\epsilon^2} e_1(Q_0)$ for $\alpha > 1$ and $\epsilon \in (0,1)$ where

$$e_1(Q_0) = \min \left\{ \frac{\int_{Q_0} \left(\left| \frac{\partial^2 v^\epsilon}{\partial {x_1}^2} \right|^2 + \left| \frac{\partial^2 v^\epsilon}{\partial {x_2}^2} \right|^2 \right) dx_1 dx_2}{\int_{Q_0} |v^\epsilon|^2 \, dx_1 dx_2} \; ; \; v^\epsilon \in H^1(Q_0), \; v^\epsilon|_{\Gamma_0} = 0 \right\}$$

is the first eigenvalue of the Laplace operator in Q_0 with homogeneous Dirichlet boundary condition in Γ_0 and Neumann everywhere else. This eigenavalue is strictly positive by hypothesis (HQ). Thus we obtain (5.4).

(b). Test functions.

The test functions we are going to construct to pass the limit in the variational formulation (5.1) are very similar to the ones we constructed in Type I thin domains. Take $\phi \in H^1(0,1)$, $\epsilon > 0$ and define the following functions in $H^1(\Omega^\epsilon)$:

$$\varphi^\epsilon(x_1, x_2) = \begin{cases} X_n^\epsilon(x_1, x_2), & (x_1, x_2) \in \Omega_+^\epsilon \cap Q_n^\epsilon, \quad n = 1, 2, \ldots, N^\epsilon = \frac{1}{L\epsilon^\alpha} \\ \phi(x_1), & (x_1, x_2) \in \Omega_- \end{cases} \tag{5.10}$$

where Q_n^ϵ is the rectangle

$$Q_n^\epsilon = (nL\epsilon^\alpha, (n+1)L\epsilon^\alpha) \times (0, G)$$

and the function X_n^ϵ is the solution of the problem

$$\begin{cases} -\dfrac{\partial^2 X^\epsilon}{\partial x_1^2} - \dfrac{1}{\epsilon^2} \dfrac{\partial^2 X^\epsilon}{\partial x_2^2} = 0, & \text{in } Q_n^\epsilon \\ \dfrac{\partial X^\epsilon}{\partial N^\epsilon} = 0, & \text{on } \partial Q_n^\epsilon \backslash \Gamma_n^\epsilon \\ X^\epsilon(x_1, x_2) = \phi(x_1), & \text{on } \Gamma_n^\epsilon \end{cases} \tag{5.11}$$

where Γ_n^ϵ is the base of the rectangle, that is,

$$\Gamma_n^\epsilon = (nL\epsilon^\alpha, (n+1)L\epsilon^\alpha) \cap \partial Q_0.$$

As we showed in the previous section, we have using Lemma 3.1,

$$\left\| \frac{\partial^2 \varphi^\epsilon}{\partial x_1^2} \right\|_{L^2(\Omega_+^\epsilon)}^2 + \frac{1}{\epsilon^2} \left\| \frac{\partial^2 \varphi^\epsilon}{\partial x_2^2} \right\|_{L^2(\Omega_+^\epsilon)}^2 \leq C \, \epsilon^{\alpha - 1} \, \|\phi'\|_{L^2(0,1)}^2 \tag{5.12}$$

which implies that

$$\|\varphi^\epsilon\|_{L^2(\Omega_+^\epsilon)}^2 + \left\| \frac{\partial^2 \varphi^\epsilon}{\partial x_1^2} \right\|_{L^2(\Omega_+^\epsilon)}^2 + \frac{1}{\epsilon^2} \left\| \frac{\partial^2 \varphi^\epsilon}{\partial x_2^2} \right\|_{L^2(\Omega_+^\epsilon)}^2 \leq C. \tag{5.13}$$

Moreover, we can show that

$$\|\varphi^\epsilon - \phi\|_{L^2(\Omega^\epsilon)} \to 0 \text{ as } \epsilon \to 0. \tag{5.14}$$

We can argue as in (5.4). If it were not true, then there will exists a $c_0 > 0$ and a sequence (that we still denote it by ϵ) such that $\|\varphi^\epsilon - \phi\|_{L^2(\Omega^\epsilon)} \geq c_0$. But then, if we define $w^\epsilon = \varphi^\epsilon - \phi$, we will have that

$$J(w^\epsilon) = \frac{\left\| \dfrac{\partial w^\epsilon}{\partial x_1} \right\|_{L^2(\Omega_+^\epsilon)}^2 + \dfrac{1}{\epsilon^2} \left\| \dfrac{\partial w^\epsilon}{\partial x_2} \right\|_{L^2(\Omega_+^\epsilon)}^2 + \|w^\epsilon\|_{L^2(\Omega_+^\epsilon)}^2}{\|w^\epsilon\|_{L^2(\Omega_+^\epsilon)}^2} \leq \frac{C}{c_0} = \tilde{C}$$

but with the same steps as we did in (a) this will contradict the fact that $e_1(Q_0) > 0$.

(c). Pass to the limit.

Now we can pass to the limit in the variational formulation (5.1). First, we note that the convergences of

$$\int_{\Omega_+^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi^\epsilon}{\partial x_2} \Big\} dx_1 dx_2 \to 0 \tag{5.15}$$

for $\epsilon \to 0$ follows from (5.12) and can be obtained as in (4.32).

Also, from (5.3) and since $\varphi^\epsilon \equiv \phi$ in Ω_-, we easily get

$$\int_{\Omega_-} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi^\epsilon}{\partial x_2} \Big\} dx_1 dx_2 \to \int_0^1 b(x_1)\, u_0'(x_1)\, \phi'(x_1)\, dx_1 \text{ as } \epsilon \to 0. \tag{5.16}$$

Let us consider now the following technical result.

Lemma 5.1. *We have*

$$\int_{S^\epsilon(x_1)} dx_2 \to q(x_1) \equiv \frac{|Q_0|}{L} + b(x_1) \qquad w^* - L^\infty(0,1).$$

Proof. If we denote by χ the characteristic function of the measurable open set Q_0, extended periodically with respect to the first variable, we have by the Average Theorem that

$$\begin{aligned} \int_{S^\epsilon(x_1)} dx_2 &= \int_0^G \chi(x_1/\epsilon^\alpha, x_2)\, dx_2 + \int_{-b(x_1)}^0 dx_2 \\ &\rightharpoonup \int_0^G \left(\frac{1}{L} \int_0^L \chi(s, x_2)\, ds \right) dx_2 + b(x_1) \qquad w^* - L^\infty(0,1) \\ &= \frac{|Q_0|}{L} + b(x_1) \quad \forall x_1 \in (0,1). \end{aligned}$$

□

Moreover,

$$\int_{\Omega^\epsilon} u^\epsilon \, \varphi^\epsilon \, dx_1 dx_2 = \int_{\Omega^\epsilon} (u^\epsilon - u_0)\; \varphi^\epsilon \, dx_1 dx_2 + \int_{\Omega^\epsilon} u_0 \; (\varphi^\epsilon - \phi)\; dx_1 dx_2 + \int_{\Omega^\epsilon} u_0 \, \phi \, dx_1 dx_2.$$

and the first two integrasl go to 0 since $\|u^\epsilon - u^0\|_{L^2(\Omega^\epsilon)} \to 0$ and $\|\varphi^\epsilon - \phi\|_{L^2(\Omega^\epsilon)} \to 0$. The last integral satisfies,

$$\int_{\Omega^\epsilon} u_0(x_1)\, \phi(x_1)\, dx_1 dx_2 = \int_0^1 u_0(x_1)\, \phi(x_1) \left(\int_{S^\epsilon(x_1)} dx_2 \right) dx_1, \to \int_0^1 q(x_1) u_0(x_1)\, \phi(x_1) dx_1$$

where we have used Lemma 5.1.

Finally, we have

$$\int_{\Omega^\epsilon} f^\epsilon \, \varphi^\epsilon \, dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \; (\varphi^\epsilon - \phi)\; dx_1 dx_2 + \int_{\Omega^\epsilon} f^\epsilon \, \phi \, dx_1 dx_2$$

but the first integral goes to 0. Moreover, with the hypothesis of the theorem, we get for the second integral

$$\int_{\Omega^\epsilon} f^\epsilon \, \phi \, dx_1 dx_2 = \int_0^1 \left(\int_{S^\epsilon(x_1)} f^\epsilon(x_1, x_2)\, dx_2 \right) \phi(x_1)\, dx_1 \to \int_0^1 \hat{f}(x_1)\, \phi(x_1)\, dx_1.$$

Therefore, we obtain from the estimates above that

$$\int_0^1 \{ b(x_1)\, u_0'(x_1)\, \phi'(x_1) + q(x_1)\, u_0(x_1)\, \phi(x_1) \}\, dx_1 = \int_0^1 \hat{f}(x_1)\, \phi(x_1)\, dx_1 \quad \forall \phi \in H^1(0,1). \tag{5.17}$$

Since this problem has a unique solution, then we obtain that the sequence $\{u^\epsilon\}_{\epsilon > 0}$ is convergent and converges to the unique solution u_0 of (5.17).

□

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